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BÄCKLUND TRANSFORMATIONS FOR GELFAND-DICKEY FLOWS, REVISITED

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1. INTRODUCTION

The j -th flow of GD_n -hierarchy (cf. [5]) is the following evolution equation

$$\frac{\partial L}{\partial t_j} = [(L^{\frac{j}{n}})_+, L] \quad (1.1)$$

on the space

$$\mathcal{D}_n = \{L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1} \mid u_i \in C^\infty(\mathbb{R}, \mathbb{C}), 1 \leq i \leq n-1\},$$

where $(L^{\frac{j}{n}})_+$ is the differential operator component of the pseudo-differential operator $L^{\frac{j}{n}}$, $\partial = \partial_x$, and $j \not\equiv 0 \pmod{n}$.

The $A_{n-1}^{(1)}$ -KdV hierarchy constructed by Drinfeld-Sokolov in [6] is the GD_n -hierarchy with matrix valued Lax pairs. The $n \times n$ KdV hierarchy constructed by the first author and Uhlenbeck in [8] is equivalent to the GD_n -hierarchy. The $n \times n$ mKdV hierarchy given in [6] and the KW-hierarchy in [7] are equivalent. Moreover, the GD_n and the $n \times n$ mKdV hierarchies are related by the Miura transform.

Bäcklund transformations (BTs) for the KdV (i.e., GD_2) hierarchy constructed from a system of compatible differential equations are well-known (cf. [1], [12]).

Adler in [2] used the Miura transform to construct a BT for (1.1) as follows: Suppose $L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1}$ is a solution of the j -th GD_n flow (1.1), and L is factored as a product of first order operators

$$L = (\partial - q_n) \cdots (\partial - q_1)$$

such that $q = \sum_{i=1}^{n-1} q_i e_{ii}$ is a solution of the $n \times n$ mKdV flow. Let \tilde{L} be defined by

$$\tilde{L} := (\partial - q_{n-1}) \cdots (\partial - q_1)(\partial - q_n).$$

Then \tilde{L} is a solution of (1.1). Rational solutions are obtained by applying Adler's BT to the vacuum solution $L = \partial^n$ (cf. [2]).

Note that operators \tilde{L} and L in Adler's result are related by

$$\tilde{L} = (\partial - q_n)^{-1} L (\partial - q_n).$$

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Given a solution L of (1.1), it is not difficult to see that $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1) if and only if h satisfies a system of ordinary differential equations in x and t variables. One goal of this paper is to find all solutions of this system for h . To achieve this, we need to use Bäcklund transformations for the $n \times n$ KdV hierarchy constructed in [8] and the equivalence of the $n \times n$ KdV and the GD_n hierarchies.

Assume $L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{(i-1)}$ is a solution of (1.1). We prove the following results:

- There exists a system for $h : \mathbb{R}^2 \rightarrow \mathbb{C}$,

$$\begin{cases} h_x^{(n)} = r_n(u, h), \\ h_t = \eta_{n,j}(u, h), \end{cases} \quad (1.2)$$

for some differential polynomials $r_n(u, h)$ and $\eta_{n,j}(u, h)$ such that $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1) if and only if h is a solution of (1.2).

- There exists a differential polynomial $\xi_n(u, h)$ such that $r_n(u, h) = (\xi_n(u, h))_x$.
- If h is a solution of (1.2) then there exists a constant $k \in \mathbb{C}$ such that h satisfies

$$(\text{BT})_{u,k} \begin{cases} h_x^{(n-1)} = \xi_n(u, h) - k, \\ h_t = \eta_{n,j}(u, h). \end{cases} \quad (1.3)$$

- Systems (1.2) and (1.3) are solvable for h .
- $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1) if and only if there exists a constant $k \in \mathbb{C}$ such that h is a solution of $(\text{BT})_{u,k}$ (we use $h * L$ to denote \tilde{L} and call $L \mapsto h * L$ a BT of L with parameter k).
- There exist differential polynomials $p_{j,1}(u, \lambda), \dots, p_{j,n}(u, \lambda)$ such that the following linear system for $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$,

$$\begin{cases} L\phi = \lambda\phi, \\ \phi_t = \sum_{i=1}^n p_{j,i}(u, \lambda)\phi_x^{(i-1)}, \end{cases} \quad (1.4)$$

is solvable for all parameter $\lambda \in \mathbb{C}$, where $p_{j,i}(u, \lambda)$ has degree $[\frac{j}{n}]$ in λ . Moreover, let $\phi_1, \dots, \phi_{n-1}$ be linearly independent solutions of (1.4) with $\lambda = k$, and $W(\phi_1, \dots, \phi_{n-1})$ the Wronskian of $\phi_1, \dots, \phi_{n-1}$. Then

$$h = (\ln W(\phi_1, \dots, \phi_{n-1}))_x$$

is a solution of $(\text{BT})_{u,k}$ (1.3) and $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1). In fact, this construction gives all solutions of (1.3).

We give algorithms to compute differential polynomials $r_n, \eta_{n,j}, \xi_n$, and $p_{j,i}$'s. For example, the second GD_3 flow for $L = \partial^3 - u_2\partial - u_1$ is (3.15), (1.2) is (3.18), $(\text{BT})_{u,k}$ is (3.17), and if h is a solution of (3.17) then $\tilde{L} = (\partial + h)^{-1}L(\partial + h) = \partial^3 - \tilde{u}_2\partial - \tilde{u}_1$ is a new solution of (3.15), where \tilde{u}_i is given by the formula (3.19). System (1.4) is (7.5).

Note that given a solution of (1.1) and a constant k , system (1.3) gives rise to $(n-1)$ -parameter family of new solutions of (1.1). When we apply BT with constant $k=0$ and $k \neq 0$ to the vacuum solution $L = \partial^n$ we obtain explicit rational solutions and soliton solutions respectively.

We also obtain the following results:

- (a) We give a permutability formula for our BTs of the GD_n flows.
- (b) Adler's BT is our BT with parameter $k=0$.
- (c) We construct a simple and natural action of the order n cyclic group \mathbb{Z}_n on the space of solutions of the KW-hierarchy and show that this \mathbb{Z}_n -action gives Adler's BT for the GD_n -hierarchy under the Miura transform.

This paper is organized as follows: We set up notations and review the constructions and relations of various KdV-type hierarchies in section 2, construct BTs for the $A_{n-1}^{(1)}$ -KdV hierarchy in section 3, and give a Permutability formula for these BTs and a relation between BTs and scaling transform in section 4. We use BTs to construct explicit rational and soliton solutions for the j -th $A_{n-1}^{(1)}$ -KdV flow in section 5, and give a natural \mathbb{Z}_n -action on the KW-hierarchy and show that Adler's BT arises from this action in section 6. In the last section, we construct BTs for the GD_n -hierarchy and explain the relation between our BTs and Adler's BTs.

2. VARIOUS KdV TYPE HIERARCHIES

In this section, we set up notations and give a brief review of the constructions of the $A_{n-1}^{(1)}$ -KdV, the $n \times n$ mKdV, the $n \times n$ KdV, and the KW-hierarchies.

Let \mathcal{B}_n^+ , \mathcal{B}_n^- , \mathcal{T}_n , \mathcal{N}_n^+ , and \mathcal{N}_n^- denote the sub-algebras of upper triangular, lower triangular, diagonal, strictly upper triangular, and strictly lower triangular matrices in $sl(n, \mathbb{C})$ respectively, and N_n^+ the subgroup of upper triangular matrices in $SL(n, \mathbb{C})$ with 1 on the diagonal entries. Let

$$\begin{aligned} \mathcal{L} &= \left\{ \xi(\lambda) = \sum_{i \leq m_0} \xi_i \lambda^i \mid \xi_i \in sl(n, \mathbb{C}), m_0 \text{ an integer} \right\}, \\ \mathcal{L}_+ &= \{ \xi(\lambda) = \sum_{i \geq 0} \xi_i \lambda^i \in \mathcal{L} \}, \\ \mathcal{L}_- &= \{ \xi(\lambda) = \sum_{i < 0} \xi_i \lambda^i \in \mathcal{L} \}. \end{aligned}$$

Note that \mathcal{L}_+ , \mathcal{L}_- are Lie subalgebras of \mathcal{L} and $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ as a direct sum of linear subspaces (such pair $(\mathcal{L}_+, \mathcal{L}_-)$ is called a *splitting* of \mathcal{L}). Given $\xi = \sum_i \xi_i \lambda^i \in \mathcal{L}$, we will use ξ_{\pm} to denote the projection of ξ onto \mathcal{L}_{\pm} w.r.t.

$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$, i.e.,

$$\xi_+ = \sum_{i \geq 0} \xi_i \lambda^i, \quad \xi_- = \sum_{i < 0} \xi_i \lambda^i.$$

2.1. The $A_{n-1}^{(1)}$ -KdV hierarchy [6]

Let

$$J = e_{1n} \lambda + b, \quad b = \sum_{i=1}^{n-1} e_{i+1,i}. \quad (2.1)$$

Given $u \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, a direct computation shows (cf. [9]) that there exists a unique $Q(u, \lambda) \in \mathcal{L}$ such that $Q(0, \lambda) = J$ and

$$\begin{cases} [\partial_x + J + u, Q(u, \lambda)] = 0, \\ Q(u, \lambda) \text{ is conjugate to } J. \end{cases} \quad (2.2)$$

Write

$$Q^j(u, \lambda) = \sum_{i \leq [\frac{j}{n}] + 1} Q_{j,i}(u) \lambda^i. \quad (2.3)$$

Let

$$V_n = \oplus_{i=1}^{n-1} \mathbb{C} e_{in}. \quad (2.4)$$

It was proved in [6] that given $u \in C^\infty(\mathbb{R}, V_n)$ and $j \not\equiv 0 \pmod{n}$, there exists a unique \mathcal{N}_n^+ -valued differential polynomial $\eta_j(u)$ such that

$$[\partial_x + b + u, Q_{j,0}(u) + \eta_j(u)] \in C^\infty(\mathbb{R}, V_n).$$

Let

$$Z_j(u, \lambda) = (Q^j(u, \lambda))_+ + \eta_j(u) = \sum_{i=0}^{[\frac{j}{n}] + 1} Z_{j,i}(u) \lambda^i. \quad (2.5)$$

So we have

$$Z_{j,i}(u) = \begin{cases} Q_{j,i}(u), & i > 0, \\ Q_{j,0}(u) + \eta_j(u), & i = 0. \end{cases}$$

The j -th flow in the $A_{n-1}^{(1)}$ -KdV hierarchy is the following evolution equation on $C^\infty(\mathbb{R}, V_n)$,

$$u_{t_j} = [\partial_x + b + u, Z_{j,0}(u)], \quad (2.6)$$

where $b = \sum_{i=1}^{n-1} e_{i+1,i}$. Moreover, it was proved in [6] that $u : \mathbb{R}^2 \rightarrow V_n$ is a solution of (2.6) if and only if

$$[\partial_x + J + u, \partial_{t_j} + Z_j(u, \lambda)] = 0 \quad (2.7)$$

for all parameter $\lambda \in \mathbb{C}$. We call (2.7) the *Lax pair of the solution u of the j -th $A_{n-1}^{(1)}$ -KdV flow*.

Definition 2.2. Frame

Let \mathcal{O} be a connected open subset of \mathbb{C} . A map $E : \mathbb{R}^2 \times \mathcal{O} \rightarrow GL(n, \mathbb{C})$ is called a *frame* of the solution u of the j -th $A_{n-1}^{(1)}$ -KdV flow if $E(x, t, \lambda)$ is holomorphic for $\lambda \in \mathcal{O}$ and satisfies

$$\begin{cases} E^{-1}E_x = J + u, \\ E^{-1}E_t = Z_j(u, \lambda), \end{cases}$$

where J is defined by (2.1). In other words, $E(\cdot, \cdot, \lambda)$ is a parallel frame for the Lax pair (2.7) of u . When u is a real solution, we also require $\overline{\mathcal{O}} = \mathcal{O}$ and E satisfies the following reality condition,

$$\overline{E(x, t, \bar{\lambda})} = E(x, t, \lambda).$$

Since $Q(0, \lambda) = J$, $Z_j(0, \lambda) = J^j$. So $u(x, t) = 0$ is a solution (the *vacuum solution*) of (2.6) and $E(x, t, \lambda) = \exp(xJ + tJ^j)$ is the frame of $u = 0$ satisfying $E(0, 0, \lambda) = I_n$ (the *vacuum frame*).

The following Theorem was proved by Drinfeld-Sokolov.

Theorem 2.3. ([6]) $u = \sum_{i=1}^{n-1} u_i e_{in}$ is a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6) if and only if $L_u = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1}$ is a solution of the j -th GD_n flow.

Hence the GD_n and the $A_{n-1}^{(1)}$ -KdV hierarchies are the same, and (2.7) is a matrix valued Lax pair for the GD_n -hierarchy.

2.4. The $n \times n$ mKdV hierarchy ([6])

The Drinfeld-Sokolov $n \times n$ mKdV hierarchy is constructed from the following splitting of the loop algebra \mathcal{L} ,

$$\begin{aligned} \mathcal{W}_+ &= \{\xi(\lambda) = \xi_0 + \sum_{i>0} \xi_i \lambda^i \in \mathcal{L} \mid \xi_0 \in \mathcal{B}_n^-\}, \\ \mathcal{W}_- &= \{\xi(\lambda) = \xi_0 + \sum_{i<0} \xi_i \lambda^i \in \mathcal{L} \mid \xi_0 \in \mathcal{N}_n^+\}. \end{aligned}$$

Let π_{bn} denote the projection of $sl(n, \mathbb{C})$ onto \mathcal{B}_n^- with respect to $sl(n, \mathbb{C}) = \mathcal{B}_n^- \oplus \mathcal{N}_n^+$, and let π_+ denote the projection of \mathcal{L} onto \mathcal{W}_+ with respect to the direct sum $\mathcal{L} = \mathcal{W}_+ \oplus \mathcal{W}_-$ of linear subspaces. Then

$$\pi_+ \left(\sum_{i \leq m_0} \xi_i \lambda^i \right) = \sum_{i>0} \xi_i \lambda^i + \pi_{bn}(\xi_0). \quad (2.8)$$

It was proved in [6] that if $q \in C^\infty(\mathbb{R}, \mathcal{T}_n)$, then $[\partial_x + b + q, \pi_{bn}(Q_{j,0}(q))]$ lies in $C^\infty(\mathbb{R}, \mathcal{T}_n)$. The j -th $n \times n$ mKdV flow is the following flow on $C^\infty(\mathbb{R}, \mathcal{T}_n)$,

$$q_{t_j} = [\partial_x + b + q, \pi_{bn}(Q_{j,0}(q))], \quad (2.9)$$

where $Q_{j,0}(u)$ is defined by (2.3). Moreover, $q : \mathbb{R}^2 \rightarrow \mathcal{T}_n$ is a solution of (2.9) if and only if

$$[\partial_x + J + q, \partial_{t_j} + \pi_+(Q^j(q, \lambda))] = 0$$

for all $\lambda \in \mathbb{C}$, where π_+ is the projection of \mathcal{L} onto \mathcal{W}_+ defined by (2.8) and $Q(q, \lambda)$ is defined by (2.2).

2.5. The $n \times n$ KdV hierarchy ([8], [9])

The $n \times n$ KdV hierarchy is constructed from an unusual splitting of \mathcal{L} . Let $\alpha = e^{\frac{2\pi i}{n}}$, and

$$\Lambda = \sum_{k=1}^{n-1} \frac{1 - \alpha^k}{1 - \alpha} e_{k, k+1}. \quad (2.10)$$

Then $\{\Lambda^i b^{n-1} \Lambda^j \mid 0 \leq i, j \leq n-1\}$ is a basis of $sl(n, \mathbb{C})$ and $\{\Lambda^i b^{n-1} \Lambda^j \mid i+j < n-1\}$ is a basis of \mathcal{N}_n^- . Let $B : sl(n, \mathbb{C}) \rightarrow \mathcal{N}_n^+$ be the linear map defined by

$$B(\Lambda^i b^{n-1} \Lambda^j) = \Lambda^i b^t \Lambda^j.$$

Note that $\text{Ker}(B) = \mathcal{B}_n^+$. It was proved in [9] that

$$\mathcal{L}_-^B = \{\xi(\lambda) = B(\xi_{-1}) + \sum_{i < 0} \xi_i \lambda^i \in \mathcal{L}\}$$

is a subalgebra of \mathcal{L} , and

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-^B$$

is a direct sum of linear subspaces. Let π_B denote the projection of \mathcal{L} onto \mathcal{L}_+ with respect to the direct sum $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-^B$. The j -th $n \times n$ KdV flow is the following evolution equation on $C^\infty(\mathbb{R}, \oplus_{i=1}^{n-1} \mathbb{C} \Lambda^i)$,

$$\xi_{t_j} = [\partial_x + b + \xi, Q_{j,0}(\xi) - B(Q_{j,-1}(\xi))], \quad (2.11)$$

where $Q_{j,i}(\xi)$ is defined by (2.3). Moreover, $\xi : \mathbb{R}^2 \rightarrow \oplus_{i=1}^n \mathbb{C} \Lambda^i$ is a solution of (2.11) if and only if

$$[\partial_x + J + \xi, \partial_{t_j} + \pi_B(Q^j(\xi, \lambda))] = 0 \quad (2.12)$$

for all parameter $\lambda \in \mathbb{C}$, where $Q(\xi, \lambda)$ is defined by (2.2).

The $n \times n$ KdV hierarchy can be also constructed from the following equivalent splitting: Let $f_i = (1 + \alpha + \cdots + \alpha^{i-1})^{-1} \Lambda^i$, where $\alpha = e^{\frac{2\pi i}{n}}$ and Λ is as in (2.10). Let $\phi_n(z) = I + \sum_{i=1}^{n-1} f_i z^i$, $\lambda = z^n$, and $T : \mathcal{L} \rightarrow \mathcal{L}$ the algebra homomorphism defined by:

$$T(A(z)) = \phi_n(z) A(z^n) \phi_n(z)^{-1}. \quad (2.13)$$

Let $\mathcal{L}_T = T(\mathcal{L})$. It was proved in [9] that

$$(\mathcal{L}_T)_+ := T(\mathcal{L}_+) = \mathcal{L}_T \cap \mathcal{L}_+, \quad (\mathcal{L}_T)_- := T(\mathcal{L}_-^B) = \mathcal{L}_T \cap \mathcal{L}_-,$$

and

$$T(e_{1n} z^n + b) = a z + b,$$

where $a = \text{diag}(1, \alpha, \dots, \alpha^{n-1})$. Hence the Lax pair (2.12) becomes

$$[\partial_x + a z + b + \xi, \partial_{t_j} + (\tilde{Q}^j(\xi, z))_+] = 0, \quad (2.14)$$

where $\tilde{Q}(\xi, z) = \phi_n(z) Q(\xi, z^n) \phi_n(z)^{-1}$. We call (2.12) and (2.14) the *Lax pair* for ξ in λ -gauge and z -gauge respectively, where $\lambda = z^n$.

We need the following Proposition (proved in [6]) to state the equivalence of the $n \times n$ KdV and the $A_{n-1}^{(1)}$ -KdV hierarchies.

Proposition 2.6. ([6]) *Given $v \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, then there exist a unique N_n^+ -valued differential polynomial Δ and V_n -valued differential polynomial u in v such that*

$$\Delta(\partial_x + J + v)\Delta^{-1} = \partial_x + J + u, \quad (2.15)$$

where V_n is defined by (2.4).

Definition 2.7. Let $\Psi : C^\infty(\mathbb{R}, \mathcal{B}_n^+) \rightarrow C^\infty(\mathbb{R}, V_n)$ and $\Gamma : C^\infty(\mathbb{R}, \mathcal{B}_n^+) \rightarrow C^\infty(\mathbb{R}, N_n^+)$ be the maps defined by

$$\Psi(v) = u, \quad \Gamma(v) = \Delta$$

respectively, where v , u , and Δ are related by (2.15).

Let

$$\mathcal{G}_j = \begin{cases} \oplus_{i=1}^{n-j} \mathbb{C}e_{i,i+j}, & j \geq 0, \\ \oplus_{i=1-j}^n \mathbb{C}e_{i,i+j}, & j < 0. \end{cases}$$

Theorem 2.8. ([10]) *Let ξ be a solution of the j -th $n \times n$ KdV flow (2.11), $u(\cdot, t) = \Psi(\xi(\cdot, t))$, and $\Delta(\cdot, t) = \Gamma(\xi(\cdot, t))$, where Ψ and Γ are the maps defined in Definition 2.7. Then we have the following results:*

- (a) u is a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6).
- (b) The map Ψ maps the space of solutions of (2.11) bijectively to the space of solutions of (2.6).
- (c) Write $\Delta = I_n + \sum_{i=1}^{n-1} \Delta_i$ with $\Delta_i \in \mathcal{G}_i$, then $\Delta_1 = 0$.
- (d) If F is a frame of a solution ξ of (2.11), then $E = F\Delta^{-1}$ is a frame of the solution $u(\cdot, t)$ of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6).

2.9. The KW-hierarchy ([7])

Let

$$\tau = e_{21} + e_{32} + \cdots + e_{n,n-1} + e_{1n},$$

$$\alpha = e^{\frac{2\pi i}{n}},$$

$$a = \text{diag}(1, \alpha, \dots, \alpha^{n-1}).$$

The KW-hierarchy is obtained from the splitting \mathcal{L}_\pm^{KW} of the algebra \mathcal{L}^{KW} , where

$$\mathcal{L}^{KW} = \{A \in \mathcal{L} \mid A(\alpha^{-1}z) = \tau A(z)\tau^{-1}\},$$

$$\mathcal{L}_\pm^{KW} = \mathcal{L}^{KW} \cap \mathcal{L}_\pm.$$

Given $v = (v_1, \dots, v_{n-1})$, let

$$P(v) = \sum_{i=1}^{n-1} v_i \tau^i. \quad (2.16)$$

Let $\hat{Q}(v, z) = \sum_{i \leq 1} \hat{Q}_{1,i}(v) z^i$ be the unique solution of

$$[\partial_x + az + P(v), \hat{Q}(v, z)] = 0$$

satisfying $\hat{Q}(0, z) = az$ and $\hat{Q}(v, z)$ is conjugate to az . Write

$$\hat{Q}^j(v, z) = \sum_{i \leq j} \hat{Q}_{j,i}(v) z^i.$$

The j -th flow in the KW-hierarchy is the following evolution equation on $C^\infty(\mathbb{R}, \oplus_{i=1}^{n-1} \mathbb{C} \tau^i)$,

$$\frac{\partial P(v)}{\partial t_j} = \sum_{i=1}^{n-1} (v_i)_{t_j} \tau^i = [\partial_x + P(v), \hat{Q}_{j,0}(v)]. \quad (2.17)$$

Moreover, v is a solution of (2.17) if and only if

$$[\partial_x + az + P(v), \partial_{t_j} + (\hat{Q}^j(v, z))_+] = 0.$$

Next we review relations between the KW, the $n \times n$ mKdV, and the $A_{n-1}^{(1)}$ -KdV hierarchies.

Theorem 2.10. ([10]) *Let $\Phi : \mathbb{R}^{1 \times (n-1)} \rightarrow \mathcal{T}_n$ be the linear isomorphism defined by $\Phi(v_1, \dots, v_{n-1}) = q = \text{diag}(q_1, \dots, q_n)$, where $q_i = \sum_{k=1}^{n-1} \alpha^{k(i-1)} v_k$ for $1 \leq i \leq n$ and $\alpha = \exp(2\pi i/n)$. Then we have the following.*

- (i) v is a solution of the j -th KW flow (2.17) if and only if $q = \Phi(v)$ is a solution of the j -th $n \times n$ mKdV flow (2.9).
- (ii) Let $V(z) = ((\alpha^{(i-1)} z)^{j-1})$, $\lambda = z^n$ and $J = e_{1n} \lambda + b$ as in (2.1). Then

$$V(z)^{-1}(\partial_x + az + P(v))V(z) = \partial_x + J + q,$$

where $P(v) = \sum_{i=1}^{n-1} v_i \tau^i$ is as defined by (2.16).

Theorem 2.11. ([6]) *Given $q = \sum_{i=1}^n q_i e_{ii} : \mathbb{R}^2 \rightarrow \mathfrak{sl}(n, \mathbb{C})$, let u_1, \dots, u_{n-1} be defined by*

$$(\partial - q_n) \cdots (\partial - q_1) = \partial^{(n)} - \sum_{i=1}^{n-1} u_i \partial^{i-1}.$$

Then we have the following.

- (i) if q is a solution of the j -th $n \times n$ mKdV-flow (2.9), then $u = \sum_{i=1}^{n-1} u_i e_{in}$ is a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6) (this is the Miura transform).
- (ii) Let Γ and Ψ be the maps defined in Definition 2.7, $\Delta(\cdot, t) = \Gamma(q(\cdot, t))$, and K a frame of the solution q of (2.9). Then $u(\cdot, t) = \Psi(q(\cdot, t))$ and $E = K \Delta^{-1}$ is a frame of the solution u of (2.6).

3. BÄCKLUND TRANSFORMATION FOR $A_{n-1}^{(1)}$ -KdV

In this section, we prove the following results:

- (i) Assume that E is a frame of a solution u of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6), and f is of the form $J + hI_n + N$ for some complex valued function h and an \mathcal{N}_n^+ -valued map N such that $\tilde{E} = Ef^{-1}$ is also a frame of a solution of (2.6). Then the entries of N are differential polynomials of u and h , and h satisfies the system (1.2).
- (ii) If u is a solution of (2.6), then system (1.2) is solvable and a solution h gives rise to a new solution of (2.6).
- (iii) All solutions of (1.2) can be constructed from frames of the Lax pair of a solution u of (2.6).
- (iv) h is a solution of (1.2) if and only if there exists a constant $k \in \mathbb{C}$ such that h is a solution of (1.3).

Bäcklund transformations for the $n \times n$ KdV hierarchy is constructed in [8] from a loop group factorization with respect to the splitting in z -gauge. We use the map T defined by (2.13) to state this result in λ -gauge as follows:

Theorem 3.1. ([8]) *Let $j \not\equiv 0 \pmod{n}$, and Λ defined by (2.10). Then there exist differential polynomials $A_j(\xi, Y)$ and $B_j(\xi, Y)$ such that if F is a frame of a solution ξ of the j -th $n \times n$ KdV flow (2.11) then $\tilde{F} = F(J + Y)^{-1}$ is a frame of a solution $\tilde{\xi}$ of (2.11) if and only if Y satisfies the following first order system,*

$$\begin{cases} Y_x = A_j(\xi, Y), \\ Y_t = B_j(\xi, Y), \end{cases} \quad (3.1)$$

where $J = e_{1n}\lambda + b$ is as in (2.1) and $Y = \sum_{i=0}^{n-1} y_i \Lambda^i$. Moreover, if ξ is a solution of (2.11), then we have the following:

- (i) System (3.1) is solvable for $Y \in C^\infty(\mathbb{R}, \oplus_{i=0}^{n-1} \mathbb{C}\Lambda^i)$.
- (ii) If Y is a solution of (3.1), then there exists a constant $k \in \mathbb{C}$ such that $\det(J + Y) = (-1)^{n-1}(\lambda - k)$.
- (iii) Given constants $k \in \mathbb{C}$ and $\mathbf{c}_0 \in \mathbb{C}^n \setminus 0$, let

$$\zeta(x, t) = (\zeta_1, \dots, \zeta_n)^t = F(x, t, k)^{-1}(\mathbf{c}_0),$$

Then there is a unique solution $Y = \sum_{i=0}^{n-1} y_i \Lambda^i$ of (3.1) satisfying $y_0 = -\frac{\zeta_{n-1}}{\zeta_n}$ and $(ke_{1n} + b + Y)\zeta = 0$.

- (iv) All solutions of (3.1) can be constructed as in (iii).

Note that if ξ is a solution of (2.11) and Y is a solution of (3.1), then

$$\tilde{\xi} = (J + Y)(J + \xi)(J + Y)^{-1} - Y_x(J + Y)^{-1} - J$$

is again a solution of (2.11). By Theorems 2.8 and 3.1, we obtain the following.

Theorem 3.2. *Let ξ be a solution of the j -th $n \times n$ -KdV flow (2.11), $Y = \sum_{i=0}^{n-1} y_i \Lambda^i$ a solution of (3.1), and $\tilde{\xi}$ the solution of (2.11) constructed from ξ and Y as in Theorem 3.1. Let $u(\cdot, t) = \Psi(\xi(\cdot, t))$, $\tilde{u} = \Psi(\tilde{\xi}(\cdot, t))$, $\Delta(\cdot, t) = \Gamma(\xi(\cdot, t))$, and $\tilde{\Delta}(\cdot, t) = \Gamma(\tilde{\xi}(\cdot, t))$, where Ψ and Γ are defined in Definition 2.7. Set*

$$f = \tilde{\Delta}(J + Y)\Delta^{-1}.$$

Then we have the following:

- (a) *Both u and \tilde{u} are solutions of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6).*
- (b) *If F is a frame of ξ , then $E = F\Delta^{-1}$ and $\tilde{E} = E f^{-1}$ are frame of solutions u and \tilde{u} of (2.6) respectively.*
- (c) *f is of the form $J + y_0 I_n + N$, where N is an \mathcal{N}_n^+ -valued map.*

Proof. From Theorem 2.8, both u and \tilde{u} are solutions of (2.6), $E = F\Delta^{-1}$ and $\tilde{E} = \tilde{F}\tilde{\Delta}^{-1}$ are frames of u and \tilde{u} respectively. It follows from a direct computation that

$$\tilde{F}\tilde{\Delta}^{-1} = F(J + Y)^{-1}\tilde{\Delta}^{-1} = E\Delta(J + Y)^{-1}\tilde{\Delta}^{-1} = E f^{-1}.$$

This proves (a) and (b).

Let $\mathcal{G}_i = \oplus_{j=1}^{n-i} \mathbb{C} e_{j,i+j}$. Write $\Delta = \sum_{i=0}^{n-1} \Delta_i$ and $\tilde{\Delta} = \sum_{i=0}^{n-1} \tilde{\Delta}_i$ with $\Delta_i, \tilde{\Delta}_i \in \mathcal{G}_i$. By Theorem 2.8, $\Delta_1 = \tilde{\Delta}_1 = 0$. So $f = J + y_0 I_n + N$ for some $N(x, t) \in \mathcal{N}_n^+$. \square

Next we prove that entries of N in Theorem 3.2 are differential polynomials of u and h . First we need a Lemma, which can be proved by a direct computation.

Lemma 3.3. *Let $u = \sum_{i=1}^{n-1} u_i e_{in}$, $E \in C^\infty(\mathbb{R}, GL(n, \mathbb{C}))$ and ϕ the first column of E . Then $E_x = E(J + u)$ if and only if $E = (\phi, \phi_x, \dots, \phi_x^{(n-1)})$ and $\phi_x^{(n)} = \sum_{i=1}^{n-1} u_i \phi_x^{(i-1)} + \lambda \phi$, where $J = e_{1n} \lambda + b$ is defined by (2.1).*

We say a differential polynomial η has order k in h if η is a polynomial in $h, h_x, \dots, h_x^{(k)}$.

Theorem 3.4. *Let E and \tilde{E} be frames of solutions $u = \sum_{i=1}^{n-1} u_i e_{in}$ and $\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_{in}$ of (2.6) respectively, and $\phi, \tilde{\phi}$ the first column of E and \tilde{E} respectively. Suppose there exists $h(x, t)$ such that $\phi = (\partial_x + h)\tilde{\phi}$. Then $E(x, \lambda) = \tilde{E}(x, \lambda)f$, where f is of the form $J + hI_n + N$, for some $N(x, t) \in \mathcal{N}_n^+$. Moreover,*

- (i) *there are differential polynomial $s_i(u, h)$ of order $(n - i)$ in h such that*

$$\tilde{u}_i = u_i + s_i(u, h), \quad 1 \leq i \leq n - 1, \quad (3.2)$$

- (ii) *the ij -th entry of N is*

$$\begin{cases} N_{ij} = C_{j-1, i-1} h_x^{(j-i)}, & 1 \leq i < j < n, \\ N_{in} = u_i + s_i(u, h) + C_{n-1, i-1} h_x^{(n-i)}, & 1 \leq i \leq n - 1, \end{cases} \quad (3.3)$$

where $C_{j,i} = \frac{j!}{i!(j-i)!}$. (We will use $f_{u,h}$ to denote such f),
 (iii) h satisfies

$$\begin{cases} h_x^{(n)} = r_n(u, h), \\ h_t = \eta_{n,j}(u, h), \end{cases} \quad (3.4)$$

for some differential polynomials $r_n(u, h)$ of order $(n-1)$ in h and $\eta_{n,j}(u, h)$ of order j in h .

Proof. From $E^{-1}E_x = J + u$, $\tilde{E}^{-1}\tilde{E}_x = J + \tilde{u}$, and $E = \tilde{E}f$, we have

$$f_x = f(J + u) - (J + \tilde{u})f. \quad (3.5)$$

Compare the nn -th entry of (3.5) to get $u_{n-1} = h_x + \tilde{u}_{n-1} + (n-1)h_x$, i.e.,

$$\tilde{u}_{n-1} = u_{n-1} - nh_x. \quad (3.6)$$

Compare the $(n-1, n)$ -entry of (3.5) to get

$$\tilde{u}_{n-2} = u_{n-2} - (u_{n-1})_x - \frac{(n-3)n}{2}h_{xx} + nh_h. \quad (3.7)$$

Use induction and compare the $(n-i, n)$ -th entry of (3.5) for $1 \leq i \leq n-1$ to see that $\tilde{u}_{n-i} = u_{n-i} + s_{n-i}(u, h)$ for some differential polynomial $s_{n-i}(u, h)$ of order i in h . This proves (i).

Let $f = \tilde{E}^{-1}E$. By assumption, the first column of f is $(h, 1, 0, \dots, 0)^t$. Lemma 3.3 implies that $\tilde{\phi}_x^{(n)} = \lambda\tilde{\phi} + \sum_{i=1}^{n-1} \tilde{u}_i\tilde{\phi}_x^{(i-1)}$. Use $\phi = \tilde{\phi}_x + h\tilde{\phi}$ to compute $\phi_x^{(i)}$ and use (3.2) to get (3.3). This proves (ii).

Compare the $1n$ -th entry of the constant term of (3.5) (as a polynomial in λ) to get $h_x^{(n)} = r_n(u, h)$ for some differential polynomial $r_n(u, h)$ of order $(n-1)$ in h . Since E and \tilde{E} are frames of u and \tilde{u} , we have $E^{-1}E_t = Z_j(u, \lambda)$ and $\tilde{E}^{-1}\tilde{E}_t = Z_j(\tilde{u}, \lambda)$, where Z_j is as in (2.5). But $E = \tilde{E}f$ implies that

$$f_t = fZ_j(u, \lambda) - Z_j(\tilde{u}, \lambda)f. \quad (3.8)$$

Compare the 11 -th entry of the constant term of (3.8) to see that $h_t = \eta_{n,j}(u, h)$ for some differential polynomial $\eta_{n,j}(u, h)$ of order j . This proves (iii). \square

The proof of Theorem 3.4 gives the following Proposition in x -variable:

Proposition 3.5. *Let $u = \sum_{i=1}^{n-1} u_i e_{in}$, $\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_{in} \in C^\infty(\mathbb{R}, V_n)$, $E, \tilde{E} \in C^\infty(\mathbb{R}, GL(n, \mathbb{C}))$ satisfying $E^{-1}E_x = J + u$ and $\tilde{E}^{-1}\tilde{E}_x = J + \tilde{u}$. Suppose $E = \tilde{E}f$ for some f of the form $J + hI_n + N$, where $h \in C^\infty(\mathbb{R}, \mathbb{C})$ and $N \in C^\infty(\mathbb{R}, \mathcal{N}_n^+)$. Then $f = f_{u,h}$, $\tilde{u}_i = u_i + s_i(u, h)$ for $1 \leq i \leq n-1$, and $h_x^{(n)} = r_n(u, h)$ as in Theorem 3.4.*

Proposition 3.6. *If $g(x) \in GL(n, \mathbb{C})$ and $\text{tr}(g^{-1}g_x) = 0$, then $\det(g(x))$ is a constant.*

Proof. Since $\frac{d}{dx} \ln(\det(g)) = \text{tr}(g^{-1}g_x)$, $\det(g)$ is constant. \square

Proposition 3.7. *Let u, \tilde{u} and E, \tilde{E} be as in Proposition 3.5. Then there exists a differential polynomial $\xi_n(u, h)$ of order $(n-2)$ in h such that*

$$\det(f_{u,h}(x, \lambda)) = (-1)^{n-1}(\lambda + h_x^{(n-1)} - \xi_n(u, h)), \quad (3.9)$$

$$h_x^{(n)} - r_n(u, h) = (h_x^{(n-1)} - \xi_n(u, h))_x, \quad (3.10)$$

where $f_{u,h}$ and $r_n(u, h)$ are as in Theorem 3.4.

Proof. Use (3.3) and a direct computation to get (3.9). Let $f_{u,h} = f$. The proof of Theorem 3.4 implies that

$$f_x - f(J+u) + (J+\tilde{u})f = (h_x^{(n)} - r_n(u, h))e_{1n}. \quad (3.11)$$

Let $w = h_x^{(n)} - r_n(u, h)$. By (3.11), we get

$$(\ln(\det f))_x = \text{tr}(f^{-1}f_x) = \text{tr}((J+u) - f^{-1}(J+\tilde{u})f + f^{-1}we_{1n}).$$

Since $\text{tr}(J+u) = \text{tr}(J+\tilde{u}) = 0$,

$$(\ln(\det f))_x = \text{tr}(f^{-1}we_{1n}) = wf^{n1},$$

where f^{n1} is the $(n1)$ -th entry of the inverse of f . By definition of f , we see that f^{n1} is equal to $(-1)^{n-1}(\det f)^{-1}$. This proves (3.10). \square

Theorem 3.8. *Suppose E, \tilde{E} are frames of solutions u, \tilde{u} of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6) respectively, and $\tilde{E} = Ef_{u,h}^{-1}$ for some smooth function h . Then there exists a constant $k \in \mathbb{C}$ such that $\det(f_{u,h}(x, t, k)) = 0$ and h satisfies*

$$(\text{BT})_{u,k} \quad \begin{cases} h_x^{(n-1)} = \xi_n(u, h) - k, \\ h_t = \eta_{n,j}(u, h), \end{cases} \quad (3.12)$$

where $\xi_n(u, h)$ is as in Proposition 3.7 and $\eta_{n,j}(u, h)$ is as in Theorem 3.4.

Proof. Recall that $\text{tr}(J+u) = \text{tr}(J+\tilde{u}) = \text{tr}(Z_j(u, \lambda)) = \text{tr}(Z_j(\tilde{u}, \lambda)) = 0$. By Proposition 3.6, determinants of both $E(x, t, \lambda)$ and $\tilde{E}(x, t, \lambda)$ are independent of x and t . Hence $\det(f_{u,h})$ is independent of x, t . But $\det(f_{u,h})$ is a degree one polynomial in λ . So there exists a constant $k \in \mathbb{C}$ such that $\det(f_{u,h}(x, t, \lambda)) = (-1)^{n-1}(\lambda - k)$. By (3.9), we have $\det(f_{u,h}) = (-1)^{n-1}(\lambda + h_x^{(n-1)} - \xi_n(u, h))$. Hence

$$h_x^{(n-1)} - \xi_n(u, h) = -k.$$

It follows from Theorem 3.4 that h satisfies the second equation of (3.12). \square

The following Theorem gives a method to construct solutions of $(\text{BT})_{u,k}$ from frames of u .

Theorem 3.9. *Let E be a frame of a solution $u = \sum_{i=1}^{n-1} u_i e_{in}$ of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6), $k \in \mathbb{C}$, $\mathbf{c}_0 \in \mathbb{C}^n \setminus 0$ constants, and $v(x, t) = (v_1, \dots, v_n)^t(x, t) := E(x, t, k)^{-1}(\mathbf{c}_0)$. If $v_n \neq 0$, then*

$$(i) \quad h = -\frac{v_{n-1}}{v_n} \text{ is a solution of } (\text{BT})_{u,k},$$

- (ii) $\tilde{E} = Ef_{u,h}^{-1}$ is a frame of a new solution $\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_{in}$ of (2.6), where \tilde{u}_i 's are given by (3.2),
- (iii) $f_{u,h}(x, t, k)(v(x, t)) = 0$ for all x, t .

Proof. By Theorem 2.8, there exist a unique solution ξ of the j -th $n \times n$ KdV flow (2.11) such that $\Psi(\xi(\cdot, t)) = u(\cdot, t)$ and $\Delta(\cdot, t) = \Gamma(\xi(\cdot, t)) \in N_n^+$, where Ψ and Γ are as in Definition 2.7. Then $F = E\Delta$ is a frame for ξ .

Let $\zeta = (\zeta_1, \dots, \zeta_n)^t = \Delta^{-1}v$, so $\zeta = F(x, t, k)^{-1}(\mathbf{c}_0)$. Let $Y(\zeta)$ be as in Theorem 3.1 (iii). Then $\tilde{F} = F(J + Y)^{-1}$ is a frame of a new solution $\tilde{\xi}$ of (2.11), $y_0 = -\frac{\zeta_{n-1}}{\zeta_n}$, and $(ke_{1n} + b + Y)\zeta = 0$. Hence $\det(J + Y)(x, t, k) = 0$. By Theorem 3.2, $\tilde{u} = \Psi(\tilde{\xi})$ is a solution of (2.6) and $\tilde{E} = Ef^{-1}$ is a frame of \tilde{u} , where $f = \tilde{\Delta}(J + Y)\Delta^{-1}$. By Theorem 3.4, $f = f_{u,y_0}$. Therefore, $\det(f_{u,y_0})(x, t, k) = 0$. By Theorem (3.8), y_0 is a solution of (3.12). Write $\Delta = I_n + \sum_{i=1}^{n-1} \Delta_i$ and $\tilde{\Delta} = I_n + \sum_{i=1}^{n-1} \tilde{\Delta}_i$ with $\Delta_i \in \mathcal{G}_i = \oplus_{j=1}^{n-i} \mathbb{C}e_{j,j+i}$. By Theorem 2.8 (b), we have $\Delta_1 = \tilde{\Delta}_1 = 0$. It follows from a direct computation that $y_0 = -\frac{\zeta_{n-1}}{\zeta_n} = -\frac{v_{n-1}}{v_n} = h$. So we have proved (i) and (ii).

From $(ke_{1n} + b + Y)(\zeta) = 0$, $\zeta = \Delta^{-1}v$ and $f_{u,h} = \tilde{\Delta}(J + Y)\Delta^{-1}$, we have $f_{u,h}(x, t, k)v(x, t, k) = 0$. \square

Theorem 3.10. *Let u be a solution of (2.6), and $k \in \mathbb{C}$ a constant. Then*

- (i) *all solutions of $(BT)_{u,k}$ are obtained by the method given in Theorem 3.9.*
- (ii) *$(BT)_{u,k}$, i.e., (3.12) with parameter k , is solvable,*

Proof. Let E be the frame of a solution u of (2.6) such that $E(0, 0, \lambda) = I_n$ for all $\lambda \in \mathbb{C}$. Let $\mathbf{c} = (c_1, \dots, c_{n-1}, 1)$ and $v = (v_1, \dots, v_n) := E^{-1}(\cdot, \cdot, k)(\mathbf{c})$. Then $v_n \neq 0$ in an open neighborhood of $(0, 0)$ in \mathbb{R}^2 . It also follows from $E^{-1}E_x = J + u$ that

$$v_x = -(e_{1n}k + b + u)v. \quad (3.13)$$

Let $h = -v_{n-1}/v_n$, and $\Phi : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ the map defined by

$$\Phi(\mathbf{c}) = (h(0, 0), h_x(0, 0), \dots, h_x^{(n-2)}(0, 0)).$$

We claim that Φ is onto. Let $\phi = -\frac{1}{v_n}v$, then $h = \phi_{n-1}$. Eq.(3.13) implies that

$$\begin{cases} \phi_x = -(ke_{1n} + b + u + hI_n)\phi, \\ \phi(0, 0) = -\mathbf{c}. \end{cases}$$

Compare the entries of both sides, we have

$$\begin{aligned} (\phi_1)_x &= -\phi_1\phi_{n-1} + k + u_1, \\ (\phi_i)_x &= -\phi_{i-1} - \phi_i\phi_{n-1} + u_i, \quad 2 \leq i \leq n-1. \end{aligned}$$

This shows that given any (c_1, \dots, c_{n-1}) there exists a solution h of (3.12) satisfying

$$h(0, 0) = -c_{n-1}, \quad h_x(0, 0) = c_{n-2} + u_{n-1}(0, 0) - c_{n-1}^2, \quad \dots$$

This proves (i). Since we have constructed solutions with all initial data, system (3.12) with parameter k is solvable. \square

Corollary 3.11. *If $u = \sum_{i=1}^{n-1} u_i e_{in}$ is a solution of (2.6) and h is a solution of $(\text{BT})_{u,k}$ (3.12), then $\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_{in}$ defined by $\tilde{u}_i = u_i + s_i(u, h)$ as in (3.2) is a solution of (2.6). Moreover, if E is a frame of u , then $E f_{u,h}^{-1}$ is a frame of \tilde{u} for $\lambda \neq k$ and $\det(f_{u,h}(x, t, \lambda)) = (-1)^{n-1}(\lambda - k)$.*

Remark 3.12. Given a solution u of (2.6), we can construct new solutions of (2.6) either by solving the compatible non-linear system $(\text{BT})_{u,k}$, i.e., (3.12), for h or solve the frame $E(x, t, k)$ by solving a linear system. When $n = 2$ this is the classical relation between two component linear systems and systems of Riccati equations.

It follows from (3.10) that if h is a solution of (3.4) then $h_x^{(n-1)} - \xi_n(u, h) = k(t)$ for some function of t . We need the following lemma to prove that $k(t)$ is a constant.

Lemma 3.13. *Let $K : \oplus_{i=0}^{n-1} \mathbb{C} \Lambda^i \rightarrow \mathbb{C}^n$ be the map defined by*

$$K(\mathbf{c}) = (y_0(0, 0), (y_0)_x(0, 0), \dots, (y_0)_x^{(n-1)}(0, 0))^t,$$

where $Y = \sum_{i=0}^{n-1} y_i \Lambda^i$ is the solution of (3.1) with initial data $Y(0, 0) = \mathbf{c}$. Then K is onto.

Proof. Let F be a frame of $\xi = \sum_{i=1}^{n-1} \xi_i \Lambda^i$ of the j -th $n \times n$ KdV flow (2.11). By Theorem 3.1, $\tilde{F} = F(J + Y)^{-1}$ is a frame for a new solution $\tilde{\xi} = \sum_{i=1}^{n-1} \tilde{\xi}_i \Lambda^i$. Hence we have

$$(J + Y)_x = (J + Y)(J + \xi) - (J + \tilde{\xi})(J + Y).$$

The coefficients of λ of the both sides are zero. So the above equality is equivalent to

$$(b + Y)_x = (b + Y)(b + \xi) - (b + \tilde{\xi})(b + Y). \quad (3.14)$$

Compare the (n, n) -th entry of both sides to get

$$y_1 = \alpha(y_0)_x + \xi_1, \quad \text{where } \alpha = e^{\frac{2\pi i}{n}}.$$

Let $\mathcal{G}_i = \oplus_{j=1}^{n-i} \mathbb{C} e_{i,j+i}$. Compare the \mathcal{G}_i -components of both sides of (3.14) for $1 \leq i \leq n-2$, and use the fact that

$$\Lambda^k b - b \Lambda^k = \left(\sum_{i=1}^{k-1} \alpha^i \right) a \Lambda^{k-1}, \quad a = \text{diag}(1, \alpha, \dots, \alpha^{n-1}), \quad \alpha = e^{\frac{2\pi i}{n}}.$$

We obtain

$$(y_i)_x \Lambda^i = (y_{i+1} - \xi_{i+1}) \left(\sum_{j=1}^i \alpha^j \right) a \Lambda^i + (\xi_{i+1} - \tilde{\xi}_{i+1}) \Lambda^{i+1} b \\ + \left(\sum_{k=0}^i y_k (\xi_{i-k} - \tilde{\xi}_{i-k}) \right) \Lambda^i, \quad (\xi_0 = \tilde{\xi}_0 = 0).$$

Compare the $(n-i, n)$ -th entry of both sides and by induction to get

$$y_{i+1} = \frac{\alpha^{i+1}}{\sum_{j=1}^i \alpha^j} (y_i)_x + p_i(y_0, \dots, y_{i-1}, \xi), \quad 1 \leq i \leq n-2,$$

where p_i 's are polynomials in y_0, \dots, y_{i-1} and ξ . Therefore, given $(y_0)_x^{(i)}(0, 0)$ for $0 \leq i \leq n-1$, we can write down $y_i(0, 0)$, $0 \leq i \leq n-1$ uniquely. So K is onto. \square

Theorem 3.14. *Let u be a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6), and h a solution of (3.4). Let $\tilde{u}_i = u_i + s_i(u, h)$ as in (3.2), and $\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_{in}$. Then \tilde{u} is a solution of (2.6) and $\tilde{E} = E f_{u,h}^{-1}$ is a frame of \tilde{u} .*

Proof. By Theorem 2.8, there exist a unique solution ξ of the j -th $n \times n$ KdV flow (2.11) such that $\Psi(\xi(\cdot, t)) = u(\cdot, t)$ and $\Delta(\cdot, t) = \Gamma(\xi(\cdot, t)) \in N_n^+$ and $F = E\Delta$ is a frame for ξ .

By Lemma 3.13, there exists $\mathbf{c} \in \oplus_{i=0}^{n-1} \mathbb{C} \Lambda^i$ such that

$$K(\mathbf{c}) = (h(0, 0), h_x(0, 0), \dots, h_x^{(n-1)}(0, 0))^t.$$

Let $Y = \sum_{i=0}^{n-1} y_i \Lambda^i$ be the solution of (3.1) with initial data

$$\sum_{i=0}^{n-1} y_i(0, 0) \Lambda^i = \mathbf{c}.$$

By Theorem 3.1, $\hat{F} = F(J+Y)^{-1}$ is a frame of the new solution $\hat{\xi}$ of (2.11). Let $\hat{u}(\cdot, t) = \Psi(\hat{\xi}(\cdot, t))$, and $\hat{\Delta}(\cdot, t) = \Gamma(\hat{\xi}(\cdot, t))$. By Theorem 2.8 again, \hat{u} is a solution of (2.6) and $\hat{E} = \hat{F}\hat{\Delta}^{-1}$ is a frame of \hat{u} . Since $F = E\hat{\Delta}$, we see that

$$\hat{E} = E\Delta(J+Y)^{-1}\hat{\Delta}^{-1} = E g^{-1},$$

where $g = \hat{\Delta}(J+Y)\Delta^{-1}$. By Theorem 3.2, g is of the form $J + y_0 I_n + N$ for some \mathcal{N}_n^+ -valued map. From Theorem 3.4, $g = f_{u, y_0}$. This shows that \hat{E}, E are frames of solutions u, \hat{u} of (2.6) and $\hat{E} = E f_{u, y_0}^{-1}$. By Theorem 3.4, y_0 satisfies (3.4). By assumption, h is a solution of (3.4). We have chosen the initial data of Y for system (3.1) so that $(y_0)_x^{(i)}(0, 0) = h_x^{(i)}(0, 0)$ for $0 \leq i \leq n-1$. So h and y_0 are solutions of (3.4) with the same initial values at $(0, 0)$. By Frobenius Theorem, there is only one solution with the same initial data. Hence $y_0 = h$. This proves $\hat{E} = E f_{u, h}^{-1}$ is a frame of the new solution $\hat{u} = \sum_{i=1}^{n-1} \hat{u}_i e_{in}$ and $\hat{u}_i = u_i + s_i(u, h)$. \square

Corollary 3.15. *Let E be a frame of a solution u of (2.6). Then $\tilde{E} = Ef_{u,h}^{-1}$ is a frame of some solution of (2.6) if and only if h is a solution of (3.4).*

Corollary 3.16. *If u is a solution of (2.6) and h is a solution of (3.4), then there exists a constant $k \in \mathbb{C}$ such that h is a solution of $(\text{BT})_{u,k}$ (3.12).*

Corollary 3.17. *If u is a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6), then*

- (a) *system (3.4) is solvable,*
- (b) *the space of solutions of system (3.4) is the union of the spaces of solutions of $(\text{BT})_{u,k}$, i.e., (3.12), for all $k \in \mathbb{C}$.*

Proof. From the proof of Theorem 3.14, given any constant $(d_0, \dots, d_{n-1})^t \in \mathbb{C}^n$, there is a solution h of (3.4) such that $h_x^{(i)}(0, 0) = d_i$ for $0 \leq i \leq n-1$. This proves (a). It follows from Theorem 3.8 and 3.14 that, if h is a solution of (3.4), then there exists a constant $k \in \mathbb{C}$ such that $h_x^{(n-1)} = \xi_n(u, h) - k$. Hence h is a solution of (3.12) with parameter k . \square

Definition 3.18. Let u be a solution of (2.6), and h a solution of $(\text{BT})_{u,k}$, i.e., (3.12). We use $h * u$ to denote the solution \tilde{u} constructed from u and h as in Corollary 3.11 and call $u \mapsto h * u$ a *Bäcklund transformation with parameter k* .

Example 3.19. We use the algorithm given in section 2 to compute the second $A_2^{(1)}$ -KdV flow and obtain

$$\begin{cases} (u_1)_t = (u_1)_{xx} - \frac{2}{3}(u_2)_{xxx} + \frac{2}{3}u_2(u_2)_x, \\ (u_2)_t = -(u_2)_{xx} + 2(u_1)_x. \end{cases} \quad (3.15)$$

Note that solutions of (3.15) give rise to solutions of the *Boussinesq equation*. In fact, take the derivative with respect to t on both sides of the second equation and use the first equation to see that u_2 satisfies the Boussinesq equation:

$$(u_2)_{tt} = -\frac{1}{3}(u_2)_x^{(4)} + \frac{4}{3}(u_2)_x^2 + \frac{4}{3}u_2(u_2)_{xx}. \quad (3.16)$$

Note that the right hand side is equal to $-\frac{1}{3}(u_2)_{xxx} + \frac{4}{3}(u_2(u_2)_x)_x$.

We use the proof of Theorem 3.4 to compute differential polynomials $s_i(u, h)$, $r_n(u, h)$ and $f_{u,h}$. The system (3.12) for a solution $u = u_1 e_{13} + u_2 e_{23}$ of (3.15) is

$$\begin{cases} h_{xx} = -u_1 + (u_2)_x + hu_2 - 3hh_x - h^3 - k, \\ h_t = \frac{2}{3}(u_2)_x - h_{xx} - 2hh_x, \end{cases} \quad (3.17)$$

and system (3.4) is

$$\begin{cases} h_x^{(3)} = (-u_1 + (u_2)_x + hu_2 - 3hh_x - h^3)_x, \\ h_t = \frac{2}{3}(u_2)_x - h_{xx} - 2hh_x. \end{cases} \quad (3.18)$$

The new solution $h * u = \tilde{u}_1 e_{13} + \tilde{u}_2 e_{23}$ of (3.15) is given by

$$\begin{cases} \tilde{u}_1 = u_1 - (u_2)_x + 3hh_x, \\ \tilde{u}_2 = u_2 - 3h_x. \end{cases} \quad (3.19)$$

Moreover, if E is a frame for u , then $Ef_{u,h}^{-1}$ is a frame for $h * u$, where

$$f_{u,h}(x, t, \lambda) = e_{13}\lambda + \begin{pmatrix} h & h_x & u_1 - (u_2)_x + h_{xx} + 3hh_x \\ 1 & h & u_2 - h_x \\ 0 & 1 & h \end{pmatrix}.$$

If u is a solution of (2.6) and h is a solution of $(\text{BT})_{u,k}$, then by (3.9), $\det(f_{u,h}(x, t, k)) = 0$. So $Ef_{u,h}^{-1}$ is not holomorphic at $\lambda = k$. However, we can multiply $Ef_{u,h}^{-1}$ on the left by some $C(\lambda)$ independent of x, t to get a new frame for $h * u$ that is holomorphic at $\lambda = k$.

Theorem 3.20. *Suppose $E(x, t, \lambda)$ is a frame of a solution u of (2.6) that is holomorphic for λ in an open subset \mathcal{O} of \mathbb{C} . Let $k \in \mathcal{O}$ be a constant, and h the solution of (3.12) constructed from $E(x, t, k)$ and $\mathbf{c} = (c_1, \dots, c_n)^t \in \mathbb{C}^n \setminus 0$ as in Theorem 3.9. Then*

- (i) $\tilde{E} = Ef_{u,h}^{-1}$ is a frame of the new solution $h * u$ and \tilde{E} is holomorphic for all $\lambda \in \mathcal{O} \setminus \{k\}$.
- (iv) Let $C(\lambda) = e_{1n}(\lambda - k) + b + \sum_{i=1}^{n-1} c_i e_{i+1,n}$. Then

$$\hat{E}(x, t, \lambda) = C(\lambda)E(x, t, \lambda)f_{u,h}(x, t, \lambda)^{-1}$$

is a frame for $h * u$ that is holomorphic for all $\lambda \in \mathcal{O}$.

Proof. By Corollary 3.17, h is a solution of (3.4). It follows from Theorem 3.14 that $h * u$ is a solution of (2.6) and $Ef_{u,h}^{-1}$ is a frame of $h * u$. By definition, $f_{u,h}(x, t, \lambda)$ is holomorphic for all $\lambda \in \mathcal{O}$. Since $\det(f_{u,h}(x, t, \lambda)) = (-1)^{n-1}(\lambda - k)$, if $\lambda \neq k$ then $Ef_{u,h}^{-1}$ is holomorphic at λ .

Let $f = f_{u,h}$, and f^\sharp the matrix whose ij -th entry is $(-1)^{i+j}\det(M_{ji})$, where M_{ij} is the $(n-1) \times (n-1)$ matrix obtained from f by crossing out the i -th row and j -th column. Then $ff^\sharp = (-1)^{n-1}(\lambda - k)\mathbf{I}_n$. So we have

$$\text{Im}(f^\sharp(x, t, k)) \subset \text{Ker}(f(x, t, k)) = \mathbb{C}E^{-1}(x, t, k)(\mathbf{c}). \quad (3.20)$$

Let

$$\begin{aligned} F(x, t, \lambda) &= C(\lambda)E(x, t, \lambda)f_{u,h}^{-1}(x, t, \lambda) \\ &= (-1)^{n-1} \frac{1}{\lambda - k} C(\lambda)E(x, t, \lambda)f^\sharp(x, t, \lambda). \end{aligned}$$

Then $F(x, t, \lambda)$ is holomorphic for $\lambda \in \mathcal{O} \setminus \{k\}$ and has a possible simple pole at $\lambda = k$. We claim that the residue of $F(x, t, \lambda)$ at $\lambda = k$ is zero. The residue of $F(x, t, \lambda)$ at $\lambda = k$ is equal to $(-1)^{n-1}C(k)E(x, t, k)f^\sharp(x, t, k)$. By (3.20), it is equal to $\phi(x, t)C(k)(\mathbf{c})$ for some function ϕ . A direct computation implies that $C(k)(\mathbf{c}) = 0$. This proves the claim and F is holomorphic at $\lambda = k$. \square

The above Theorem will be used to construct Bäcklund transformations for the n-d central affine curve flow in [11].

4. PERMUTABILITY AND SCALING TRANSFORM

In this section, we

- (i) give a Permutability formula for Bäcklund transformations,
- (ii) prove that the conjugation of a Bäcklund transformation with parameter $k = 1$ by the r -scaling transform gives a BT with parameter $k = r^{-n}$

for the $A_{n-1}^{(1)}$ -KdV hierarchy.

Theorem 4.1. [Permutability for BT]

Let u be a solution of (2.6), $k_1, k_2 \in \mathbb{C}$ constants, h_i solutions of $(BT)_{u, k_i}$ (3.12), and $h_i * u$ the solution of (2.6) construct from u and h_i for $i = 1, 2$. Suppose $h_1 \neq h_2$. Set

$$\begin{cases} \tilde{h}_1 = h_1 + \frac{(h_1 - h_2)_x}{h_1 - h_2}, \\ \tilde{h}_2 = h_2 + \frac{(h_1 - h_2)_x}{h_1 - h_2}. \end{cases} \quad (4.1)$$

Then

- (i) \tilde{h}_1 is a solution of $(BT)_{h_2 * u, k_1}$ and \tilde{h}_2 is a solution of $(BT)_{h_1 * u, k_2}$,
- (ii) $\tilde{h}_1 * (h_2 * u) = \tilde{h}_2 * (h_1 * u)$.

Proof. Let $E(x, t, \lambda)$ be a frame of u . By Theorem 3.10, there exist constant vectors $v_1^0, v_2^0 \in \mathbb{R}^n$ that give h_1, h_2 , i.e.,

$$\begin{aligned} v_1 &= (v_{1,1}, \dots, v_{1,n})^t = E(\cdot, \cdot, k_1)^{-1} v_1^0, \\ v_2 &= (v_{2,1}, \dots, v_{2,n})^t = E(\cdot, \cdot, k_2)^{-1} v_2^0, \\ h_i &= -\frac{v_{i,n-1}}{v_{i,n}}, \quad i = 1, 2. \end{aligned}$$

From (3.13),

$$(v_i)_x = -E^{-1} E_x(x, t, k_i) v_i = -(J + u) |_{\lambda=k_i} v_i, \quad i = 1, 2.$$

In particular, we have

$$\begin{cases} (v_{i,n-1})_x = -v_{i,n-2} - u_{n-1} v_{i,n}, \\ (v_{i,n})_x = -v_{i,n-1}, \end{cases} \quad i = 1, 2.$$

Therefore,

$$(h_i)_x = \frac{v_{i,n-2}}{v_{i,n}} - h_i^2 + u_{n-1}, \quad i = 1, 2. \quad (4.2)$$

By Theorem 3.9, $E_i(x, t, \lambda) = E(x, t, \lambda) f_{u, h_i}^{-1}$ is a frame for $h_i * u$ for $i = 1, 2$ respectively. Let

$$\tilde{v}_1 = E_2^{-1}(x, t, k_1) v_1^0, \quad \tilde{v}_2 = E_1(x, t, k_2)^{-1} v_2^0.$$

Now we compute

$$\tilde{h}_i = -\frac{\tilde{v}_{i,n-1}}{\tilde{v}_{i,n}}.$$

From

$$\tilde{v}_1 = E_2^{-1}(x, t, k_1)v_1^0 = f_{u,h_2}(k_1)E(x, t, k_1)^{-1}v_1^0 = f_{u,h_2}(k_1)v_1,$$

we get

$$\begin{cases} \tilde{v}_{1,n-1} = v_{1,n-2} + h_2v_{1,n-1} + (u_{n-1} - (h_2)_x)v_{1,n}, \\ \tilde{v}_{1,n} = v_{1,n-1} + h_2v_{1,n}. \end{cases}$$

Together with (4.2), we have

$$\tilde{h}_1 = -h_2 + \frac{v_{1,n-2}v_{2,n} - v_{2,n-2}v_{1,n}}{v_{2,n-1}v_{1,n} - v_{1,n-1}v_{2,n}} = h_1 + \frac{(h_1 - h_2)_x}{h_1 - h_2}.$$

Similarly, $\tilde{h}_2 = h_2 + \frac{(h_1 - h_2)_x}{h_1 - h_2}$.

A direct computation implies $f_{u_2, \tilde{h}_1}f_{u, h_2} = f_{u_1, \tilde{h}_2}f_{u, h_1}$. Hence $\tilde{h}_2 * (h_1 * u) = \tilde{h}_1 * (h_2 * u)$. \square

Next we review the *scaling transform* of the $A_{n-1}^{(1)}$ -KdV hierarchy.

Proposition 4.2. ([3]) *Let $u = \sum_{i=1}^{n-1} u_i e_{in}$ be a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow, and $r \in \mathbb{R} \setminus \{0\}$. Let $D(r) = \text{diag}(1, r, \dots, r^{n-1})$, and*

$$(r \cdot u_i)(x, t) = r^{n+1-i} u_i(rx, r^j t), \quad 1 \leq i \leq n-1.$$

Then

- (i) $(r \cdot u) = \sum_{i=1}^{n-1} (r \cdot u_i) e_{in}$ *is a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow,*
- (ii) *if $E(x, t, \lambda)$ is a frame of u then*

$$\tilde{E}(x, t, \lambda) := D(r)^{-1} E(rx, r^j t, r^{-n} \lambda) D(r)$$

is a frame of $r \cdot u$.

So the multiplicative group $\mathbb{R} \setminus \{0\}$ acts on the space of solutions of the j -th $A_{n-1}^{(1)}$ -KdV flow. The following Theorem proves that the conjugation of BT with parameter $k = 1$ by a scaling transform gives BT with arbitrary non-zero real parameter.

Theorem 4.3. *Let u be a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow, $r \in \mathbb{R} \setminus \{0\}$, and h a solution of $(\text{BT})_{r \cdot u, 1}$. Then*

- (i) $\hat{h}(x, t) = r^{-1} h(r^{-1} x, r^{-j} t)$ *is a solution of $(\text{BT})_{u, r^{-n}}$,*
- (ii) $r^{-1} \cdot (h * (r \cdot u)) = \hat{h} * u$.

Proof. Let E be the frame of u with $E(0, 0, \lambda) = I_n$. By Proposition 3.6,

$$\det E(x, t, \lambda) = 1. \tag{4.3}$$

From Proposition 4.2,

$$F(x, t, \lambda) = D(r)^{-1} E(rx, r^j t, r^{-n} \lambda) D(r)$$

is a frame for $\tilde{u} = r \cdot u$, where $D(r) = \text{diag}(1, r, \dots, r^{n-1})$. It follows from Theorem 3.14 that

$$F_1(x, t, \lambda) = F(x, t, \lambda) f_{r \cdot u, h}^{-1}(x, t, \lambda)$$

is a frame for $h * (r \cdot u)$. Apply the scaling transform given by r^{-1} to $h * \tilde{u}$, we have

$$\begin{aligned} F_2(x, t, \lambda) &= D(r) F_1(r^{-1}x, r^{-j}t, r^n\lambda) D(r)^{-1} \\ &= E(x, t, \lambda) D(r) f_{\tilde{u}, h}^{-1}(r^{-1}x, r^{-j}t, r^n\lambda) D(r)^{-1}. \end{aligned} \quad (4.4)$$

is a frame for $\hat{u} = r^{-1} \cdot (h * (r \cdot u))$. A direct computation implies that

$$\begin{aligned} D(r) f_{\tilde{u}, h}(r^{-1}x, r^{-j}t, r^n\lambda) D(r)^{-1} &= \\ r(e_{1n}\lambda + b + r^{-1}h(r^{-1}x, r^{-j}t)I_n + r^{-1}D(r)ND(r)^{-1}), \end{aligned}$$

where $f_{\tilde{u}, h} = e_{1n}\lambda + b + hI_n + N$ and N is strictly upper triangular.

So we have proved $F_2(x, t, \lambda) = r^{-1}E(x, t, \lambda)g^{-1}(x, t, \lambda)$ is a frame of $\hat{u} = r^{-1} \cdot (h * (r \cdot u))$, where

$$g(x, t, \lambda) = e_{1n}\lambda + b + \hat{h}(x, t)I_n + r^{-1}D(r)N(r^{-1}x, r^{-j}t)D(r)^{-1}.$$

Note that $r^{-1}D(r)N(r^{-1}x, r^{-j}t)D(r)^{-1} \in \mathcal{N}_n^+$, and

$$\hat{E}(x, t, \lambda) = rF_2(x, t, \lambda) = E(x, t, \lambda)g^{-1}(x, t, \lambda) \quad (4.5)$$

is a frame of \hat{u} . So $g = f_{u, \hat{h}}$.

By Theorem 3.8, we have $\det(f_{\tilde{u}, h}) = (-1)^{n-1}(\lambda - 1)$. It follows from (4.3), (4.4) and (4.5) that

$$\begin{aligned} \det(f_{u, \hat{h}}^{-1}) &= \det(\hat{E}(x, t, \lambda)) = r^n \det(F_2(x, t, \lambda)) \\ &= r^n \det(f_{\tilde{u}, h}^{-1}(r^{-1}x, r^{-j}t, r^n\lambda)) \\ &= (-1)^{n-1} \frac{r^n}{r^n\lambda - 1} = \frac{(-1)^{n-1}}{\lambda - r^{-n}}. \end{aligned}$$

Hence $\det(f_{u, \hat{h}}) = (-1)^{n-1}(\lambda - r^{-n})$. \square

5. EXPLICIT SOLUTIONS

In this section we apply Theorem 3.20 to $u = 0$ to construct explicit solutions of (2.6).

Proposition 5.1. *If we apply BTs with non-zero parameters to the vacuum solution $u = 0$ of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6) repeatedly, then we obtain infinitely many families of explicit solutions of (2.6) that are rational functions of exponential functions.*

Proof. Let $\lambda = z^n$, $\alpha = e^{\frac{2\pi i}{n}}$,

$$\begin{aligned} D(z) &= \text{diag}(1, z, \dots, z^{n-1}), \\ \Xi &= (\alpha^{(i-1)(j-1)}), \\ A_i(x, t, z) &= \exp(\alpha^{i-1}zx + (\alpha^{i-1}z)^jt), \\ (m_1(x, t, z), \dots, m_n(x, t, z)) &= (e^{A_1}, \dots, e^{A_n})\Xi. \end{aligned}$$

A direct computation implies that the vacuum frame is

$$E(x, t, z^n) = e^{xJ+tJ^j} = \frac{1}{n}D(z)^{-1} \begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ m_n & m_1 & \cdots & m_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_2 & m_3 & \cdots & m_1 \end{pmatrix} D(z).$$

Given constants $k \neq 0$ and $\mathbf{c} = (c_1, \dots, c_n)^t \in \mathbb{C}^n$, let

$$(v_1, v_2, \dots, v_n)^t = E(x, t, k^n)^{-1}(\mathbf{c}), \quad h = -\frac{v_{n-1}}{v_n}. \quad (5.1)$$

Note that

$$E(x, t, k^n)^{-1} = E(-x, -t, k^n).$$

Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{C}^{n \times 1}$, then

$$\begin{aligned} (m_3, \dots, m_n, m_1, m_2) &= (e^{A_1}, \dots, e^{A_n})\Xi\pi_1, \\ (m_2, m_3, \dots, m_n, m_1) &= (e^{A_1}, \dots, e^{A_n})\Xi\pi_2. \end{aligned}$$

where $\pi_1 = (e_3, \dots, e_n, e_1, e_2)$, $\pi_2 = (e_2, e_3, \dots, e_n, e_1)$ (permutation matrices). Hence

$$\begin{aligned} v_{n-1}(x, t) &= k^{-(n-2)}(e^{A_1(-x, -t, k)}, \dots, e^{A_n(-x, -t, k)})\Xi\pi_1 D(k)\mathbf{c}, \\ v_n(x, t) &= k^{-(n-1)}(e^{A_1(-x, -t, k)}, \dots, e^{A_n(-x, -t, k)})\Xi\pi_2 D(k)\mathbf{c}. \end{aligned}$$

This implies that h given by (5.1) is a rational function of exponentials. Hence entries of the new solution $h * 0$ are rational functions of exponential functions. It follows from Theorem 3.20 that, the entries of the frame of $h * 0$,

$$\tilde{E}(x, t, \lambda) = C(\lambda)E(x, t, \lambda)f_{u, h}(x, t, \lambda),$$

are given by rational functions of exponentials, where $C(\lambda) = e_{1n}(\lambda - k) + b + \sum_{i=1}^{n-1} c_i e_{i+1, n}$. If we apply Theorem 3.20 to $h * 0$ with parameter $k_2 \neq 0$, then we get another family of new solutions that are rational of exponential functions. \square

Example 5.2 (Explicit solutions of the j -th $A_{n-1}^{(1)}$ -KdV flow).

We use the same notation as in the proof of Proposition 5.1. First note that $\Xi\pi_2 D(k)$ is invertible. So given any $\mathbf{a} \in \mathbb{C}^{n \times 1}$, we can find suitable $\mathbf{c} \in \mathbb{C}^n$ such that $\Xi\pi_2 D(k)\mathbf{c} = \mathbf{a}$. We will write down some explicit solutions for the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6) below.

Choose \mathbf{c} such that $\Xi\pi_2 D(k)\mathbf{c} = e_1 + e_2$. A simple computation implies that $\Xi\pi_1 D(k)\mathbf{c} = e_1 + \alpha e_2$. The function h defined by (5.1) is

$$h = -k \frac{e^{-(kx+k^j t)} + \alpha e^{-(\alpha kx + \alpha^j k^j t)}}{e^{-(kx+k^j t)} + e^{-(\alpha kx + \alpha^j k^j t)}}. \quad (5.2)$$

This is a solution of (3.12) with parameter $k \in \mathbb{C}$. Let A, B be defined by

$$\begin{cases} A + B = -(kx + k^j t), \\ A - B = -(\alpha kx + \alpha^j k^j t). \end{cases}$$

Then (5.2) becomes $h = -k \frac{e^{A+B} + \alpha e^{A-B}}{e^{A+B} + e^{A-B}}$, so we get

$$h = \frac{k}{2} ((1 - \alpha) \tanh B - (1 + \alpha)), \quad (5.3)$$

where $\alpha = e^{\frac{2\pi i}{n}}$ and $B = \frac{1}{2}((1 - \alpha)kx + (1 - \alpha^j)k^j t)$. Note that h defined by (5.3) is a solution of (3.12) with parameter k .

(i) If $k = (1 + \bar{\alpha})\mu$ for some $\mu \in \mathbb{R}$, then h given by (5.3) becomes a real function,

$$h = -\mu \sin\left(\frac{2\pi}{n}\right) \tan \theta - (1 + \cos\left(\frac{2\pi}{n}\right))\mu, \quad (5.4)$$

where $\theta = \mu \sin\left(\frac{2\pi}{n}\right)x + \sum_{i=0}^{j-1} C_{j,i} \mu^j \sin\left(\frac{2(j-i)\pi}{n}\right)t$. Hence

$$h * 0 = 3h_x(he_{13} - e_{23})$$

is a real solution of the j -th $A_{n-1}^{(1)}$ -KdV flow.

In particular, when $j = 2$, (5.4) becomes

$$h = -\mu \sin \gamma \tan(\mu \sin \gamma (x + 2(1 + \cos \gamma)\mu t)) - (1 + \cos \gamma)\mu, \quad (5.5)$$

where $\gamma = \frac{2\pi}{n}$, and $\mu \in \mathbb{R}$. Then $h * 0 = 3h_x(he_{13} - e_{23})$ is a real solution of the second $A_{n-1}^{(1)}$ -KdV flow.

(ii) If $n = 2m$ and j is odd, we choose \mathbf{c} such that $\Xi\pi_2 D(k)\mathbf{c} = e_1 + e_{m+1}$. Then $\Xi\pi_1 D(k)\mathbf{c} = e_1 - e_{m+1}$, and h defined by (5.1) is

$$h = k \tanh(kx + k^j t).$$

When $k \in \mathbb{R} \setminus 0$, we obtain soliton solutions for the j -th $A_{2m-1}^{(1)}$ -KdV flow. For example, when $m = 1$ and $j = 3$, this give rise to the 1-soliton solutions of the KdV equation.

(iii) If $n = 2m$ and j is even, we choose \mathbf{c} such that $\Xi\pi_2 D(k)\mathbf{c} = e_1 + e_{m+1}$. Then $\Xi\pi_1 D(k)\mathbf{c} = e_1 - e_{m+1}$, and h defined by (5.1) is

$$h = k \tanh(kx).$$

This gives stationary smooth solutions $h * 0$ for the j -th $A_{2m-1}^{(1)}$ -KdV flow.

Example 5.3 (Explicit solutions for the second $A_2^{(1)}$ -KdV flow (3.15)).

For $n = 3$, we have $\alpha = e^{\frac{2\pi i}{3}}$.

(i) Let $\frac{k}{2}(1 - \alpha) = -\mu$. Then h defined by (5.3) becomes

$$h = \sqrt{3}\mu \tanh(\sqrt{3}\mu(x - 2\mu it)) + \mu i. \quad (5.6)$$

Substitute (5.6) to (3.19) with $u = 0$ to see that

$$\begin{cases} u_1 = 9\mu^3 \operatorname{sech}^2(\sqrt{3}\mu(x - 2\mu it))(\sqrt{3} \tanh(\sqrt{3}\mu(x - 2\mu it)) + i), \\ u_2 = -9\mu^2 \operatorname{sech}^2(\sqrt{3}\mu(x - 2\mu it)) \end{cases}$$

is a complex solution of (3.15). Note that u_2 is the solution of the Boussinesq equation (3.16) obtained in [4].

(ii) h given by (5.5) is

$$h = -\frac{\sqrt{3}}{2}\mu \tan\left(\frac{\sqrt{3}}{2}\mu(x + \mu t)\right) - \frac{\mu}{2}.$$

Let $\nu = \frac{\mu}{2}$. Then

$$h_\nu = -\sqrt{3}\nu \tan(\sqrt{3}\nu(x + 2\nu t)) - \mu$$

is a real solution of (3.15) and u_1, u_2 defined by

$$\begin{cases} u_1 = 9\nu^3 \sec^2(\sqrt{3}\nu(x + 2\nu t))(1 + \sqrt{3} \tan(\sqrt{3}\nu(x + 2\nu t))), \\ u_2 = 9\nu^2 \sec^2(\sqrt{3}\nu(x + 2\nu t)), \end{cases}$$

is the real solution of (3.15).

Proposition 5.4. *If we apply BT with parameter $k = 0$ to the vacuum solution $u = 0$ of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.6) repeatedly, then we obtain infinitely many families of explicit rational solutions of (2.6)*

Proof. Note that $E(x, t, \lambda) = e^{xJ+tJ^j}$ is a frame of the solution $u = 0$ of (2.6). Since entries of $E(x, t, 0) = \exp(bx + b^j t)$ are polynomials in x and t , we see that the solution h of (3.12) constructed in Theorem 3.9 with $k = 0$ is a rational solution and $\tilde{E} = C(\lambda)E f_{0,h}^{-1}$ is a frame of the solution $h * 0$. Apply Theorem 3.20 with parameter $k = 0$ and a constant vector \mathbf{c}_1 to $h * 0$ to get another solution. But $\tilde{E}(x, t, 0) = C(0)E(x, t, 0)f_{0,h}^{-1}(x, t, 0)$ is rational in x, t . So the solution h_1 of (3.12) with $u = h * 0$ is rational. Hence $h_1 * (h * 0)$ is a rational function. \square

Example 5.5. [Rational solutions for the second $A_2^{(1)}$ -KdV flow]

The coefficients of the constant term and λ^{-1} of the frame $E(x, t, \lambda) = e^{xJ+tJ^2}$ of the vacuum solution $u = 0$ as a power series in λ are

$$E_0(x, t) = \exp(bx + b^2 t) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ \frac{x^2}{2} + t & x & 1 \end{pmatrix},$$

$$E_1(x, t) = \begin{pmatrix} xt + \frac{1}{6}x^3 & t + \frac{1}{2}x^2 & x \\ \frac{1}{2}t^2 + \frac{1}{2}x^2t + \frac{1}{24}x^4 & xt + \frac{1}{6}x^3 & t + \frac{1}{2}x^2 \\ \frac{1}{2}xt^2 + \frac{1}{6}x^3t + \frac{1}{5!}x^5 & \frac{1}{2}t^2 + \frac{1}{2}x^2t + \frac{1}{24}x^4 & xt + \frac{1}{6}x^3 \end{pmatrix}.$$

We apply Theorem 3.20 to $u = 0$ and $v_0 = (a_1, a_2, 1)^t$ to get new solutions:

- (i) $h = \frac{a_1 x - a_2}{1 + a_1(\frac{x^2}{2} - t) - a_2 x}$ is a solution of (3.18).
- (ii) $h * 0 = u_1 e_{13} + u_2 e_{23}$ is a rational solution of (3.15), where

$$\begin{cases} u_1 = -\frac{3(a_1 x - a_2)(\frac{1}{2}a_1^2 x^2 - a_1 a_2 x + a_1^2 t + a_2^2 - a_1)}{(1 + a_1(\frac{x^2}{2} - t) - a_2 x)^3}, \\ u_2 = \frac{3(\frac{1}{2}a_1^2 x^2 - a_1 a_2 x + a_1^2 t + a_2^2 - a_1)}{(1 + a_1(\frac{x^2}{2} - t) - a_2 x)^2}. \end{cases}$$

Example 5.6. [Rational solutions for the second $A_{n-1}^{(1)}$ -KdV flow]

Recall that $E(x, t, \lambda) = e^{xJ + tJ^2}$ is a frame of $u = 0$. The coefficients of the constant and λ of $E(x, t, \lambda)$ as a power series in λ are $E(x, t, 0) = \exp(bx + b^2 t)$ and $E_1(x, t) = \frac{\partial}{\partial \lambda}|_{\lambda=0} \exp(xJ + tJ^2)$. A direct computation implies that $J^k = (b^t)^{n-k} \lambda + b^k$, $J^n = \lambda I_n$, and $\frac{\partial}{\partial \lambda}|_{\lambda=0} J^k = (b^t)^{n-k}$ for $1 \leq k \leq n-1$. So entries of $E_1(x, t)$ are polynomials in x, t . In fact, for $n = 2m+1$, we get

$$\begin{aligned} E_1(x, t) = & (I + \sum_{j=1}^{n-1} \frac{x^j}{j!} b^j) (\sum_{j=1}^m \frac{t^j}{j!} (b^t)^{n-2j} + \sum_{j=1}^m \frac{t^{m+j}}{(m+j)!} b^{2j-1}) \\ & + (\sum_{j=1}^{n-1} \frac{x^j}{j!} (b^t)^j + \frac{x^n}{n!} + \sum_{j=1}^{n-1} \frac{x^{n+j}}{(n+j)!} b^j) (I + \sum_{j=1}^m \frac{t^j}{j!} b^{2j}). \end{aligned}$$

For $n = 2m$,

$$\begin{aligned} E_1(x, t) = & (I + \sum_{j=1}^{n-1} \frac{x^j}{j!} b^j) (\sum_{j=1}^{m-1} \frac{t^j}{j!} (b^t)^{n-2j} + \frac{t^m}{m!} + \sum_{j=1}^{m-1} \frac{t^{m+j}}{(m+j)!} b^{2j}) \\ & + (\sum_{j=1}^{n-1} \frac{x^j}{j!} (b^t)^j + \frac{x^n}{n!} + \sum_{j=1}^{n-1} \frac{x^{n+j}}{(n+j)!} b^j) (I + \sum_{j=1}^{m-1} \frac{t^j}{j!} b^{2j}). \end{aligned}$$

Apply Theorem 3.20 repeatedly to construct infinitely many families of explicit rational solutions.

6. Z_n -ACTION FOR THE KW FLOWS

In this section we give a natural action of the cyclic group Z_n of n elements on the space of solutions of the j -th KW flow and show that Adler's Bäcklund transformation comes from this Z_n -action.

Theorem 6.1. *Let $\alpha = e^{\frac{2\pi i}{n}}$, $a = \text{diag}(1, \alpha, \dots, \alpha^{n-1})$, and $\tau = e_{21} + e_{32} + \dots + e_{n,n-1} + e_{1n}$ as in the KW-hierarchy. If $v = (v_1, \dots, v_{n-1})$ is a solution of the j -th KW flow (2.17), then*

$$\alpha \cdot v = (\alpha v_1, \dots, \alpha^{(n-1)} v_{n-1})$$

is a solution of (2.17). In particular, this defines an Z_n -action on the space of solutions of (2.17).

Proof. Let $\mathcal{L}^{KW} = \mathcal{L}_+^{KW} \oplus \mathcal{L}_-^{KW}$ be the splitting that gives the KW-hierarchy in section 2.

First note that $a\tau = \alpha\tau a$. A direct computation implies that $\text{Ad}(a^{-1})$ leaves \mathcal{L}_\pm^{KW} invariant, i.e., $a^{-1}(\mathcal{L}_\pm^{KW})a \subset \mathcal{L}_\pm^{KW}$. So

$$a^{-1}(\partial_x + az + P(v))a = \partial_x + az + P(\alpha \cdot v).$$

Recall that $\hat{Q}(v, z)$ is defined by $[\partial_x + az + P(v), \hat{Q}(v, z)] = 0$ and $\hat{Q}(v, z)$ is conjugate to az . Hence $\hat{Q}(\alpha \cdot v, z) = a^{-1}\hat{Q}(v, z)a$, which implies that

$$(\hat{Q}^j(\alpha \cdot v, z))_+ = a^{-1}(\hat{Q}^j(v, z))_+ a. \quad (6.1)$$

Since v is a solution of (2.17), we have

$$[\partial_x + az + P(v), \partial_t + (\hat{Q}^j(v, z))_+] = 0.$$

Conjugate the above equation by a^{-1} to get

$$\begin{aligned} & a[\partial_x + az + P(v), \partial_t + (\hat{Q}^j(v, z))_+]a^{-1} \\ &= [\partial_x + az + P(\alpha \cdot v), \partial_t + a^{-1}(\hat{Q}^j(v, z))_+ a], \quad \text{by (6.1),} \\ &= [\partial_x + az + P(\alpha \cdot v), \partial_t + (\hat{Q}^j(\alpha \cdot v, z))_+] = 0. \end{aligned}$$

This proves that $\alpha \cdot v$ is a solution of (2.17). \square

Next we use Theorem 6.1 to show that Adler's BT arises naturally from the Z_n -action on the KW-flows.

Corollary 6.2. *Let $\Phi : \mathbb{R}^{1 \times (n-1)} \rightarrow \mathcal{T}_n$ be the linear isomorphism defined by $\Phi(v) = \text{diag}(q_1, \dots, q_n)$ in Theorem 2.10, where $q_i = \sum_{k=1}^{n-1} \alpha^{k(i-1)} v_k$. Let $q = \text{diag}(q_1, \dots, q_n)$ be a solution of the j -th $n \times n$ mKdV flow, and $v = \Phi^{-1}(q)$ the corresponding solution of the j -th KW flow. Then $\Phi(\alpha^{n-1} \cdot v) = \text{diag}(q_n, q_1, \dots, q_{n-1})$ and is a solution of the j -th $n \times n$ mKdV flow.*

Proof. By Theorem 2.10, v is a solution of the j -th KW flow. By Theorem 6.1, $\alpha \cdot v = (\alpha v_1, \dots, \alpha^{n-1} v_{n-1})$ is a solution of the j -th KW flow. Hence $\Phi(\alpha \cdot v)$ is also a solution of the j -th $n \times n$ mKdV flow. A simple computation implies that $\Phi(\alpha \cdot v) = \text{diag}(q_2, \dots, q_n, q_1)$. So

$$\Phi(\alpha^{n-1} \cdot v) = \text{diag}(q_n, q_1, \dots, q_{n-1})$$

is a solution of the j -th $n \times n$ mKdV flow. \square

This Corollary gives a proof of Adler's result: Suppose

$$L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{(i-1)} = (\partial - q_1) \cdots (\partial - q_n)$$

such that L is a solution of the j -th GD_n flow (1.1) and $q = \text{diag}(q_1, \dots, q_n)$ is a solution of the j -th $n \times n$ mKdV flow (2.9). By Corollary 6.2,

$$\text{diag}(q_n, q_1, \dots, q_{n-1})$$

is a solution of (2.9). It follows from Theorem 2.11 that $\tilde{L} = (\partial - q_n)(\partial - q_1) \cdots (\partial - q_{n-1})$ is a solution of (1.1).

7. RELATION BETWEEN ADLER'S BT AND OUR BT

In this section, we prove that if L is a solution of the j -th GD_n flow (1.1), then $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1) if and only if h is a solution of (3.4). We also show that Adler's BT is our BT with parameter $k = 0$.

Proposition 7.1. *Let L be a solution of the j -th GD_n flow (1.1) and $h : \mathbb{R}^2 \rightarrow \mathbb{C}$. Then $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a n -th order differential operator and is a solution of (1.1) if and only if*

$$\begin{cases} h_x = r_n(u, h), \\ h_t = w_{n,j}(u, h), \end{cases} \quad (7.1)$$

for some differential polynomial $w_{n,j}(u, h)$, where $r_n(u, h)$ is as in Theorem 3.4.

Proof. It follows from Lemma 3.3 and the proof of Theorem 3.4 that \tilde{L} is in \mathcal{D}_n if and only if h satisfies $h_x^{(n)} = r_n(u, h)$ and the coefficient \tilde{u}_i of ∂^{i-1} is given by (3.2). So \tilde{L} is a solution of (1.1) if and only if

$$\tilde{L}_t = [(\partial + h)^{-1}(L^{\frac{j}{n}})_+(\partial + h) - (\partial + h)^{-1}h_t, \tilde{L}].$$

Note that $\tilde{L}^{j/n} = (\partial + h)^{-1}L^{j/n}(\partial + h) = \partial^j + \dots$, by comparing the highest power of ∂ , we have

$$(\partial + h)^{-1}(L^{\frac{j}{n}})_+(\partial + h) - (\partial + h)^{-1}h_t = (\tilde{L}^{\frac{j}{n}})_+.$$

Hence we obtain

$$h_t = -(\tilde{L}^{\frac{j}{n}})_-(\partial + h) + (\partial + h)(\tilde{L}^{\frac{j}{n}})_-.$$

Let p_{-1} , \tilde{p}_{-1} denote the coefficient of ∂^{-1} of $L^{j/n}$ and $\tilde{L}^{j/n}$ respectively. Since coefficients of $(L^{j/n})_+$ and $(\tilde{L}^{\frac{j}{n}})_+$ are differential polynomials in u and \tilde{u} respectively, p_{-1} and \tilde{p}_{-1} are differential polynomials in u and h . So we have

$$h_t = w_{n,j}(u, h) := -p_{-1}(u, h) + \tilde{p}_{-1}(u, h). \quad (7.2)$$

Hence we proved the proposition. \square

Theorem 7.2. *Let $L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1}$ be a solution of the j -th GD_n flow (1.1), and $h \in C^\infty(\mathbb{R}^2, \mathbb{C})$. Then the following are equivalent:*

- (a) $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of the j -th GD_n flow (1.1),
- (b) h is a solution of (7.1),
- (c) h is a solution of (3.4),
- (d) h is a solution of $(BT)_{u,k}$ (3.12) for some constant $k \in \mathbb{C}$.

Proof. The equivalence of (a) and (b) is given by Proposition 7.1, and the equivalence of (c) and (d) is given by Corollary 3.17.

We prove (a) implies (c). Suppose $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1). Write $\tilde{L} = \partial^n - \sum_{i=1}^{n-1} \tilde{u}_i \partial^{i-1}$. By Theorem 2.3, both $u = \sum_{i=1}^{n-1} u_i e_{in}$ and $\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_{in}$ are solutions of (2.6). Let $\tilde{E}(x, t, \lambda)$ be a frame of

the solution \tilde{u} , and $\tilde{\phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_n)^t$ the first column of \tilde{E} . It follows from Lemma 3.3 that $\{\tilde{\phi}_1, \dots, \tilde{\phi}_n\}$ are solutions of $\tilde{L} - \lambda = 0$. Since $L - \lambda = (\partial + h)(\tilde{L} - \lambda)(\partial + h)^{-1}$, $\phi_i = (\partial + h)\tilde{\phi}_i$ is a solution of $L - \lambda = 0$ for $1 \leq i \leq n$. Let $E = ((\phi_i)_x^{(j-1)})$, and $f = \tilde{E}^{-1}E$. So $E = \tilde{E}f$. A direct computation implies that the first column of f is $(h, 1, \dots, 0)^t$ and $\det(f(x, t, \lambda))$ is of degree one in λ . Since \tilde{E} is a frame for \tilde{u} , $\det(\tilde{E})$ is independent of x, t , is an analytic function $c(\lambda)$, and $c(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$. So $\det(E(x, t, \lambda)) \neq 0$ for generic (x, t, λ) , i.e., $\{\phi_1, \dots, \phi_n\}$ is a basis of $L - \lambda = 0$ for generic λ . Hence E is a frame of u . By Theorem 3.4, $E = \tilde{E}f_{u,h}$ and h is a solution of (3.4).

Next we prove (d) implies (a). Suppose h is a solution of $(BT)_{u,k}$ (3.12). By Theorem 3.14 we have

- (i) $\tilde{u} = h * u$ is a solution of (2.6),
- (ii) if \tilde{E} is a frame of \tilde{u} , then $E = \tilde{E}f_{u,h}$ is a frame for u for all $\lambda \neq k$.
- (iii) the $i1$ -th entries of $\phi_i E$ and $\tilde{\phi}_i \tilde{E}$ are related by $\phi_i = (\partial + h)\tilde{\phi}_i$.

Assume $\lambda \neq k$. Lemma 3.3 implies that $\{\phi_1, \dots, \phi_n\}$ is a basis of $L - \lambda = 0$. This is also a basis of $(\partial + h)(\tilde{L} - \lambda)(\partial + h)^{-1} = 0$, where $\tilde{L} = \partial^n - \sum_{i=1}^{n-1} \tilde{u}_i \partial^{(i-1)}$. It follows from Lemma 3.3 that $L - \lambda = (\partial + h)(\tilde{L} - \lambda)(\partial + h)^{-1}$. Hence $L = (\partial + h)\tilde{L}(\partial + h)^{-1}$. Or equivalently, $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$. \square

Although both $\eta_{n,j}(u, h)$ and $w_{n,j}(u, h)$ can be computed, these two differential polynomials are not in explicit closed form; hence we do not know whether they are the same. But systems (7.1) and (3.4) have the same set of solutions.

Corollary 7.3. *Let $L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1}$ be a solution of the j -th GD_n flow (1.1), $u = \sum_{i=1}^{n-1} u_i e_{in}$, and $k \in \mathbb{C}$ a constant. Let $p_{j,i}(u, \lambda)$ denote the $i1$ -th entry of $Z_j(u, k)$, where $Z_j(u, \lambda)$ is defined by (2.5) for the j -th $A_{n-1}^{(1)}$ -KdV flow. Then*

$$\begin{cases} (L - k)\phi = 0, \\ \phi_t = \sum_{i=1}^n p_{j,i}(u, k)\phi_x^{(i-1)}, \end{cases} \quad (7.3)$$

is solvable for $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$, Moreover, if $\phi_1, \dots, \phi_{n-1}$ are linearly independent solutions of (7.3), then

$$h = (\ln(W(\phi_1, \dots, \phi_{n-1})))_x,$$

is a solution of $(BT)_{u,k}$ (3.12) and $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1), where $W(f_1, \dots, f_{n-1}) = \det((f_i)_x^{(j-1)})$ is the Wronskian.

Proof. By Theorem 2.3, u is a solution of (2.6). Let $g = (\phi, \phi_x, \dots, \phi_x^{(n-1)})$. Then system (7.3) is equivalent to

$$\begin{cases} g^{-1}g_x = ke_{1n} + b + u, \\ g^{-1}g_t = Z_j(u, k). \end{cases} \quad (7.4)$$

This is the Lax pair of u with $\lambda = k$, so it is solvable. If ϕ is a solution of (7.3), then g satisfies (7.4). Let $(v_1, \dots, v_n)^t = g^{-1}e_n$. It follows from Theorem 3.9 that $h = -\frac{v_{n-1}}{v_n}$ is a solution of (3.12) with parameter k . Use Cramer's rule to get the formula of h in terms of Wronskians. \square

For example, when $n = 3$, equation (7.3) is

$$\begin{cases} \phi_{xxx} - u_2\phi_x - u_1\phi = \lambda\phi, \\ \phi_t = -\frac{2}{3}u_2\phi + \phi_{xx}. \end{cases} \quad (7.5)$$

Note that \tilde{L} and L in Adler's Theorem satisfies the condition $\tilde{L} = (\partial - q_n)^{-1}L(\partial - q_n)$. So by Theorem 7.2, $-q_n$ is a solution of (1.3) for some $k \in \mathbb{C}$. Next we show that $k = 0$. In other words, Adler's BT is our BT with parameter $k = 0$.

Theorem 7.4. *Let $L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{(i-1)} = (\partial - q_n) \cdots (\partial - q_1)$ be a solution of the j -th GD_n flow (1.1) such that $q = \sum_{i=1}^n q_i e_{ii}$ is a solution of the j -th $n \times n$ mKdV flow. Then $-q_n$ is a solution of (BT) _{$u,0$} , i.e., (3.12) with parameter $k = 0$.*

Proof. Let K denote the parallel frame for the Lax pair of the solution q of the j -th $n \times n$ mKdV flow at $\lambda = 0$ with $K(0,0) = I_n$, i.e., K is the solution of

$$\begin{cases} K^{-1}K_x = b + q, \\ K^{-1}K_t = \pi_{bn}(Q_{j,0}(q)), \end{cases}$$

where $Q_{j,0}(q)$ is defined in section 2, $b = \sum_{i=1}^{n-1} e_{i+1,i}$, and π_{bn} is the projection of $sl(n, \mathbb{C})$ onto \mathcal{B}_n^- along \mathcal{N}_n^+ . Since both $b + q$ and $\pi_{bn}(Q_{j,0}(q))$ are lower triangular and $K(0,0) = I_n$, we see that $K(x,t) \in B_n^-$, i.e. lower triangular, for all x, t . Let $\phi = (\phi_1, \dots, \phi_n)^t$ denote the first column of P . It follows from $K_x = K(b + q)$ that the j -th column of K is

$$(\partial - q_{j-1}) \cdots (\partial - q_1)\phi,$$

and $L\phi = (\partial - q_n) \cdots (\partial - q_1)\phi = 0$. Since K is lower triangular, we obtain

$$(\partial - q_i) \cdots (\partial - q_1)\phi_i = 0, \quad (7.6)$$

for $1 \leq i \leq n-1$.

Let $\Delta(\cdot, t) := \Gamma(q(\cdot, t))$, where Γ is given in Definition 2.7. By Theorem 2.11, $g = K\Delta^{-1}$ is a frame of the Lax pair of the solution u of (2.6) at $\lambda = 0$. Note that Δ is in N_n^+ , so the first column of g is also ϕ . Use $g^{-1}g_x = b + u$ to obtain that $g = (\phi, \phi_x, \dots, \phi_x^{(n-1)})$.

Let

$$R := (\partial - q_{n-1}) \cdots (\partial - q_1) = \partial^{n-1} - \sum_{i=1}^{n-1} \xi_i \partial^{i-1}.$$

By (7.6), $R\phi_j = 0$ for all $1 \leq j \leq n-1$. So we have

$$\sum_{i=1}^{n-1} \xi_i \phi_j^{(i-1)} = \phi_j^{(n-1)}, \quad 1 \leq j \leq n-1.$$

By Cramer's rule,

$$\xi_{n-1} = \frac{\det(\eta, \dots, \eta_x^{(n-4)}, \eta_x^{(n-3)}, \eta_x^{(n-1)})}{\det(\eta, \eta_x, \dots, \eta_x^{(n-2)}),} \quad (7.7)$$

where $\eta = (\phi_1, \dots, \phi_{n-1})^t$. Recall that $g = (\phi, \dots, \phi_x^{(n-1)})$. Let g^{ij} denote the ij -th entry of g^{-1} . Then the numerator and denominator of the right hand side of (7.7) are $-\det(g)g^{n-1,n}$ and $\det(g)g^{nn}$ respectively. So $\xi_n = -\frac{g^{n-1,n}}{g^{nn}}$. By Theorem 3.9, ξ_{n-1} is a solution of (1.3) with $k = 0$. Compare the coefficient of ∂^{n-1} of $L = (\partial - q_n)R$ to obtain $q_n = -\xi_{n-1}$. Hence $-q_n$ is a solution of (1.3) with $k = 0$. \square

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